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The inertial damping and resonance of cellular convection in a rotating fluid annulus: steady linear theory

By P. G. DANIELS

Department of Mathematics, University College London†

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The phenomenon of steady cellular convection in a rotating fluid in which the stratification is statically unstable is well known. If the temperature field also varies in the horizontal direction a thermal wind is generated which can diminish the amplitude of the cells, or if inertial effects are strong enough, confine them to a narrow vertical band. On the other hand, in an enclosed container resonance can occur which increases the amplitude of the cells beyond the scope of a linear theory.

1. Introduction

In his study of the stability of an infinite horizontally bounded rotating fluid heated from below, Chandrasekhar (1961, chap. 2) describes how, for sufficiently high Prandtl numbers, instability first sets in at a critical value R_c of the Rayleigh number associated with a horizontal wavelength a_c in the form of steady convection cells. His analysis for general values of an Ekman number

$$E = \nu/\Omega L^2, \tag{1.1}$$

where ν is the kinematic viscosity of the fluid, Ω the angular velocity of rotation and L a typical length scale of the system, shows that when there are two free boundaries $R_c \sim E^{-\frac{4}{3}}$ and $a_c \sim E^{\frac{1}{3}}$ as $E \to 0$. He also suggested that a similar law holds when one or both boundaries are fixed and this was confirmed in the case of two rigid boundaries by Homsy & Hudson (1971), who also determined the next two terms in the expansion of R_c as $E \to 0$. One advantage of studying this limiting situation is that the convection cells are so thin (of horizontal extent $\sim E^{\frac{1}{3}}$) that they may be regarded as a local phenomenon at a given horizontal location, and this allowed Daniels & Stewartson (1977, hereafter referred to as I) to make an analytic study of their behaviour when the temperature field varies in the horizontal direction on the length scale L, thus extending the earlier work of Daniels (1976), which established the initial onset of the cellular motion. In both cases the model consisted of a rotating annulus insulated around its inner and outer curved walls and upper horizontal surface. The curved vertical walls were parallel to the axis of rotation, which was anti-parallel to the direction of gravity g. Similar configurations are widely used in laboratory studies which hope to model features of the atmospheric circulation. In many of these studies the motion has been generated by maintaining the inner and outer curved walls at different constant temperatures (see Hide & Mason 1975) but the aspect of differential heating at the same horizontal level may be of interest in relation to the circulation of the oceans (see Defant 1961, p. 492) and of the atmosphere of Venus (see, for example, de Rivas 1973).

In the steady axisymmetric study in I convective effects are measured by the parameter

$$\lambda = \sigma\beta\gamma^{\frac{4}{3}}E^{-\frac{2}{3}}, \tag{1.2}$$

† Present address: Department of Mathematics, The City University, London.

where σ is the Prandtl number, γ is the aspect ratio of the meridional cross-section of the annulus, which is assumed to have width L, and β is a thermal Rossby number

$$\beta = \alpha g \Delta T / \Omega^2 L, \tag{1.3}$$

where α is the coefficient of thermal expansion and ΔT the imposed temperature difference along the base. With $E \ll 1$, $\lambda \sim 1$ corresponds to a Rayleigh number $\sim E^{-\frac{4}{3}}$, and since the conductive interior temperature field in the annulus, being a solution of Laplace's equation, is statically stable in the inner half but unstable in the outer half of the meridional cross-section, steady convection cells are superimposed on part of this field if λ exceeds a certain critical value $\lambda_c^{(1)}$. Although these cells are essentially the same as those described by Chandrasekhar (1961, chap. 2), in the differentially heated rotating annulus they are actually an integral part of the steady solution forced by the boundary conditions at the outer side wall. For $\lambda < \lambda_c^{(1)}$ the side-wall effect is confined to an $E^{\frac{1}{3}}$-layer in which an exponentially decaying solution reduces the $O(\alpha g \Delta T E^{\frac{1}{2}} \Omega^{-1})$ component of vertical velocity in the interior (which is generated through the thermal-wind relation and Ekman pumping) to zero at the wall. However, as described in Daniels (1976), once $\lambda > \lambda_c^{(1)}$ the first mode of the linear side-wall solution, expressed as an infinite series of vertical modes, no longer decays exponentially and is instead oscillatory, with the result that convection cells of horizontal wavelength $O(E^{\frac{1}{3}})$ penetrate into the interior of the fluid. In I their development is traced using a multiple-scales technique which exploits the fact that their amplitudes vary on the much larger scale of the annulus itself. In this way it is possible to provide a complete analytic description of the cells for certain stratification profiles, including their eventual decay, which occurs in the neighbourhood of the vertical transition line at which a suitably defined local Rayleigh number has fallen to its critical value. A further increase in λ results in the penetration of successive modes of the side-wall solution, which decay in the neighbourhood of their respective critical Rayleigh numbers. In this way the basic temperature field acts as a filter of the complicated disturbances which spread from the outer wall, allowing the lowest mode to penetrate the furthest into the interior. No cells occur in the inner half of the annulus, where the stratification is stable, and the inner side-wall solution decays exponentially for all λ.

Resonance can also occur at a certain set of values of λ at which the frequency of the forced convection cells coincides with one of the natural spatial frequencies of the annulus. With no forcing these values of λ simply correspond to the existence of eigensolutions in the annulus, which with the finite side walls now occur at discrete intervals, in contrast to the corresponding continuous spectrum in the infinite problem considered by Chandrasekhar (1961). No doubt a nonlinear analysis will be required to resolve the question of how the flow develops beyond the first resonance as the instability takes over and inertial effects control the resonance, but this is beyond the scope of the present paper, in which we concentrate upon inertial aspects of the steady linear problem. Although the above argument would seem to invalidate such solutions once the first resonance has occurred, we shall show that inertial effects due to the thermal wind can not only damp the cells but completely remove the possibility of resonance for all values of λ if an appropriately scaled thermal Rossby number is sufficiently large.

The question of resonance is discussed in § 6. A precise statement of the basic equations and underlying assumptions, with a brief summary of the results of I for the case in whch β is vanishingly small, is made in § 2. Using an approximation based on a stratification field which is independent of height, it may be shown (§ 3) that, for a given mode n and values of λ in excess of an appropriate critical value $\lambda_c^{(n)}$, the structure of the cells is first affected by the vertically sheared zonal flow in the interior when $\beta \sim E^{\frac{2}{3}}$. For values of $\beta \gg E^{\frac{2}{3}}$ the amplitudes of the cells are so reduced that they are essentially confined to a narrow vertical band of width $\sim \sigma^{\frac{1}{2}}$ ($\ll 1$) at the outer wall; as the value of $\beta E^{-\frac{2}{3}}$ increases there is a shift in emphasis as the cells decrease in amplitude in the interior of the annulus, the flow in the neighbourhood of their transition lines becomes insignificant and the solution can be determined uniquely from the three boundary conditions at the outer wall alone by a boundary-layer analysis (§ 4). For values of λ within a critical distance $O(E^{\frac{1}{3}})$ of $\lambda_c^{(n)}$ the solutions of §§ 3 and 4 are invalid and must be replaced by a new solution. Here inertial effects do not influence the leading-order flow until $\beta \sim E^{\frac{1}{2}}$ and the details of the solution, described in § 5, are crucial to the determination of the first resonance of the system. Extensions of the theory, including the solution for more general stratification profiles, are discussed in § 7.

2. Formulation and assumptions

We shall assume the flow is steady and axisymmetric, that both centrifugal and curvature effects are negligible and that the fluid obeys the Oberbeck–Boussinesq approximation. Thus the equations of motion referred to axes rotating with the constant angular velocity Ω of the annulus are

$$\partial u/\partial x + \partial w/\partial z = 0, \tag{2.1a}$$

$$-2v + \beta(u\,\partial u/\partial x + w\,\partial u/\partial z) = -\partial p/\partial x + E\nabla^2 u, \tag{2.1b}$$

$$2u + \beta(u\,\partial v/\partial x + w\,\partial v/\partial z) = E\nabla^2 v, \tag{2.1c}$$

$$\beta(u\,\partial w/\partial x + w\,\partial w/\partial z) = -\partial p/\partial z + E\nabla^2 w + T, \tag{2.1d}$$

$$\lambda(u\,\partial T/\partial x + w\,\partial T/\partial z) = \gamma^{\frac{4}{3}} E^{\frac{1}{3}} \nabla^2 T. \tag{2.1e}$$

Here the origin of co-ordinates is taken at the mid-point of the lower surface of the cross-section of the annulus through its axis with the x axis radially outwards and the z axis vertically upwards. The flow then takes place in the region bounded by the planes

$$x = \pm\tfrac{1}{2}, \quad z = 0, \quad z = D/L = \gamma, \tag{2.2}$$

where D is the height of the annulus. The velocity components u, v, $w(x,z)$ (in the radial, azimuthal and vertical directions respectively), pressure $p(x,z)$ and temperature $T(x,z)$ are non-dimensional variables related to the actual physical (starred) quantities by the formulae

$$\mathbf{x} = \mathbf{x}^*/L, \quad \mathbf{u} = \Omega\mathbf{u}^*/\alpha g \Delta T, \quad T = (T^* - T_0^*)/\Delta T, \quad p = p^*/\alpha\rho_0 g L \Delta T, \tag{2.3}$$

where p^* represents the departure of the pressure from the hydrostatic pressure that prevails when the fluid is at rest at a uniform temperature T_0^* and density ρ_0. We define a stream function ψ from (2.1 a) by

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x, \tag{2.4}$$

and assume that ψ, and all the velocity components, vanish on the surfaces (2.2), which are taken as rigid and impermeable. All the upper surfaces of the annulus will be assumed to be thermally insulating while along the base the motion is forced by a monotonic radial temperature gradient of the form

$$T(x,0) = \tfrac{1}{2}\sin \pi x \quad (-\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}). \tag{2.5}$$

The problem is now completely defined in terms of the four parameters γ, E, β and λ, and we shall assume E to be small. In (2.1*e*) the Prandtl number σ has been replaced by the parameter λ defined in (1.2). This is convenient in the present study since we shall be concerned with aspects of the cellular regime which occurs when $\lambda \sim 1$.

If β is sufficiently small all the nonlinear inertial terms on the left-hand sides of (2.1*a–e*) may be neglected, as in the basic set of equations of I. However, if $\lambda \sim 1$ the second term of the left-hand side of the heat equation is significant in the Stewartson $E^{\frac{1}{3}}$-layers located along the inner and outer walls of the annulus. These layers are required to reduce the interior vertical component of velocity to zero at the walls. The interior azimuthal velocity

$$v = v_0 + \lambda E^{\frac{1}{6}}\gamma^{-\frac{1}{3}}v_1 + \lambda^2 E^{\frac{1}{3}}\gamma^{-\frac{2}{3}}v_2 + \dots, \quad v_0 = -\cos \pi x[\sinh \pi(\gamma - z) - \tfrac{1}{2}\sinh \pi\gamma]/4\cosh \pi\gamma, \tag{2.6}$$

automatically tends to zero since it is related to the conductive temperature field

$$T = T_0 + \lambda E^{\frac{1}{6}}\gamma^{-\frac{1}{3}}T_1 + \lambda^2 E^{\frac{1}{3}}\gamma^{-\frac{2}{3}}T_2 + \dots, \quad T_0 = \sin \pi x \cosh \pi(\gamma - z)/2\cosh \pi\gamma, \tag{2.7}$$

by the thermal-wind equation. Provided that λ is less than a certain critical value the solution in the side-wall layers decays into the geostrophic interior and the flow is completed by Ekman layers of thickness $\sim E^{\frac{1}{2}}$ at $z = 0, \gamma$, which complete the meridional circulation by providing the necessary radial transport of fluid. However if λ is greater than this value the side-wall solution at the outer wall becomes oscillatory, with the result that the interior stream function must be represented in the form

$$\begin{aligned}\psi &= \tfrac{1}{16}E^{\frac{1}{2}}\tanh \pi\gamma \cos \pi x + \dots \\ &\quad + \left[\gamma^{\frac{1}{3}}E^{\frac{5}{6}} \sum_{\alpha_0 > 0} D_\alpha(x,Z,E)\exp\left\{\frac{-i}{\gamma^{\frac{1}{3}}E^{\frac{1}{3}}}\int_x^{\frac{1}{2}} (\alpha_0(x) + \gamma^{-\frac{1}{3}}E^{\frac{1}{6}}\alpha_1(x))\,dx\right\} + \text{c.c.}\right],\end{aligned} \tag{2.8}$$

where the first terms represent the basic solution corresponding to (2.6) and (2.7), valid when λ is subcritical, and the final term (where c.c. denotes the complex conjugate) represents an oscillatory solution with wavelength $\sim E^{\frac{1}{3}}$ and amplitude

$$D_\alpha(x,Z,E) = D_{\alpha 0}(x,Z) + \gamma^{-\frac{1}{3}}E^{\frac{1}{6}}D_{\alpha 1}(x,Z) + \gamma^{-\frac{2}{3}}E^{\frac{1}{3}}D_{\alpha 2}(x,Z) + \dots, \tag{2.9}$$

which varies on the order-one horizontal scale of the annulus. The real eigenvalues α_0 must be determined from the ordinary differential equation and boundary conditions for $D_{\alpha 0}$, which are

$$4\frac{\partial^2 D_{\alpha 0}}{\partial Z^2} - \alpha_0^2\left(\lambda\frac{\partial T_0}{\partial z} + \alpha_0^4\right)D_{\alpha 0} = 0, \quad D_{\alpha 0}(x,0) = D_{\alpha 0}(x,1) = 0 \quad (0 \leqslant Z \leqslant 1), \tag{2.10}$$

where $z = \gamma Z$ and $\partial T_0/\partial z$ is the conductive stratification, given by the solution (2.7). Real values of α_0, corresponding to the existence of the oscillatory solutions, depend upon the fact that $\partial T_0/\partial z$ is negative and that λ is positive. For a temperature field of the form (2.7) this restricts the cells to the outer half of the annulus and for a given λ (sufficiently high) there exists a series of transition lines $x = x_n$ ($n = 1, 2, \dots; x_i < x_{i+1}$)

at which the nth pair of eigenvalues of the system (2.10) converge to a single value, no longer existing for $x < x_n$. Alternatively, if we consider the situation at a given location x (for instance at the outer side wall itself) there is a series of critical values $\lambda_c^{(n)}$ of λ at which the nth pair of eigenvalues is generated, and there are no cells at all if $\lambda < \lambda_c^{(1)}$.

Complete analytic solutions are clearly dependent upon the nature of the stratification profile, which appears in (2.10) and which is strictly given by the formula (2.7). However general properties of the cells may be obtained analytically by making the approximation that the stratification is independent of height:

$$\partial T_i/\partial z = -C_i(x) \quad (i = 0, 1, \ldots). \tag{2.11}$$

Using the first of these equations ($i = 0$) it may be shown that $\lambda_c^{(n)}$ and x_n (if it exists) are given by the formulae

$$\lambda_c^{(n)} = 4^{\frac{1}{3}}3(n\pi)^{\frac{4}{3}}/C_0(\tfrac{1}{2}), \quad \lambda C_0(x_n) = 4^{\frac{1}{3}}\,3(n\pi)^{\frac{4}{3}}. \tag{2.12}$$

The higher-order terms $(E^{\frac{1}{6}}, E^{\frac{1}{3}}, \ldots)$ in (2.8), (2.9) and (2.11) are generated by the corresponding terms in the basic solutions (2.6) and (2.7) and also by the influence of the Ekman layers, which, when matched with the leading terms in the oscillatory part of the interior solution (2.8), force the second-order functions $D_{\alpha 1}$ and α_1. Although these terms may be calculated in a straightforward manner, as in the analysis of I, and result, for instance, in order-$E^{\frac{1}{6}}$ displacements of the transition lines and order-$E^{\frac{1}{6}}$ corrections to the critical values of λ, in the following sections we shall exclude such effects, which tend to mask the crucial features of the analysis. This may be done in a consistent manner if we assume that the shear stress rather than the velocity is specified at $z = 0, \gamma$ and also that the functions $C_1, C_2, \ldots$ are subsumed in the leading-order stratification field C_0.

The two functions $D_{\alpha 0}$ corresponding to a given pair of eigenvalues α_0 satisfying the system (2.10) contain arbitrary multiplicative functions $A_\alpha(x)$ which are found from consideration of the order-$E^{\frac{1}{3}}$ term in the expansion (2.9). In the immediate neighbourhood of the side wall the stream function (2.8) involves an additional solution corresponding to the eigensolution of (2.10) with negative imaginary part which decays within a distance $\sim E^{\frac{1}{3}}$ of the wall. The complete solution for ψ of given mode n then contains five unknown constants, three of which are determined from the requirements of zero velocity and insulation at $x = \frac{1}{2}$ and the remaining two from the requirement that the cells should decay beyond the transition line at x_n. The precise manner in which this is achieved is described in I although we note that the essential feature is that the solution (2.8) becomes invalid within a distance $O(E^{\frac{2}{9}}\gamma^{\frac{2}{9}}\lambda^{-\frac{1}{6}}C_0(x_n)^{\frac{1}{6}}\,C_0'(x_n)^{-\frac{1}{3}})$ of x_n. Here the solution must be reformulated, is $O(E^{-\frac{1}{18}})$ larger than the solution (2.8) and can be written in terms of Airy functions.

3. The critical regime $\beta \sim E^{\frac{2}{3}}$ for $\lambda > \lambda_c^{(n)}$

If the value of λ is less than that required to generate the cells in the interior of the fluid, the nonlinear inertial terms in the equations of motion (2.1) may be neglected throughout the annulus if $\beta \ll 1$. However, once the cells appear they introduce radial and vertical velocity components $\sim E^{\frac{2}{3}}$ and $\sim E^{\frac{1}{2}}$ (respectively) into the interior of the fluid, so that the sizes of the inertial term due to the rotation and the dominant nonlinear inertial term, which occurs in the azimuthal component of the momentum equation, are

$$u \sim E^{\frac{5}{6}}, \quad \beta w\, \partial v/\partial z \sim \beta E^{\frac{1}{2}}\, \partial v_0/\partial z \sim \beta E^{\frac{1}{2}}, \tag{3.1}$$

where v_0 is the basic interior zonal velocity given by (2.6). Thus the cell structure will be significantly modified when the scaled thermal Rossby number $\beta_0 = \beta\gamma^{\frac{4}{3}}E^{-\frac{1}{3}} \sim 1$. The leading-order equation (2.10) for $D_{\alpha 0}$ then contains an additional term, the effect of which may be estimated by solving the equation for $\beta_0 \ll 1$. The eigenvalue equation for α_0 becomes

$$\alpha_0^6 - \lambda C_0 \alpha_0^2 - \tfrac{1}{2} i \gamma^{\frac{1}{3}} \beta_0 C_0' \alpha_0 + 4n^2\pi^2 = 0 \quad (n = 1, 2, \ldots; \beta_0 \ll 1), \tag{3.2}$$

where terms $O(\beta_0^2)$ are neglected and use of the approximate formulation (2.11) and the thermal-wind relation has led to the replacement of the zonal-velocity formula (2.6) by

$$\partial v_0/\partial z = \tfrac{1}{2}\, \partial T_0/\partial x = -\tfrac{1}{2}\gamma(Z-1)\, C_0'(x), \tag{3.3}$$

where we assume

$$C_0(\tfrac{1}{2}) = C \quad (>0), \quad C_0'(\tfrac{1}{2}) = 0, \quad C_0''(\tfrac{1}{2}) = -c \quad (<0), \quad C_0(0) = 0, \tag{3.4}$$

and $C = \frac{1}{2}$ and $c = \frac{1}{2}\pi^2$ for the particular profile (2.5).

The effect of the additional term in (3.2) is to prevent the set of roots which lie at opposite points in the third and fourth quadrants of the complex α_0 plane when $\lambda C_0 = 0$ from ever reaching the real axis for any finite value of λC_0, in contrast to the situation giving rise to (2.12). Thus provided that $\beta_0 C_0' \neq 0$ there are three roots for α_0 with $\operatorname{Im}\alpha_0 < 0$ for each value of n (the third set of roots lies along the negative imaginary axis) and if $\beta_0 C_0'(\frac{1}{2}) > 0$ it would be possible to obtain a consistent solution in and $E^{\frac{1}{3}}$-layer on the outer side wall which decays into the geostrophic interior and satisfies the three conditions at the wall. However the wall is insulated and so $C_0'(\frac{1}{2}) = 0$, but since the three decaying solutions appear as soon as $\beta_0 C_0'$ is non-zero, this result indicates that the transition of the solution must occur near $x = \frac{1}{2}$ when $\beta_0 \sim 1$, suggesting that the problem may then be solvable by a boundary-layer approach. This will be confirmed in § 4, but it is clear that the transition of the cellular regime must occur at lower values of β. The reason for this is that the leading-order solution for $D_{\alpha 0}$ depends upon the details of higher-order terms in the multiple-scales solution of comparative order $E^{\frac{1}{3}}$, since it is the compatability of the solution at this stage which determines the amplitude functions A_α. Thus inertial modifications to the cell amplitudes due to the interior zonal flow occur when

$$\beta_1 = \beta\gamma^{\frac{4}{3}}E^{-\frac{2}{3}} \sim 1, \tag{3.5}$$

and following I we use the assumptions (3.3) to obtain

$$D_{\alpha 0} = A_\alpha(x) \sin n\pi Z, \tag{3.6}$$

where

$$A_\alpha(x) = K_n \alpha_0^{-\beta_1/2\lambda} |\alpha_0^4 - \tfrac{1}{3}\lambda C_0|^{-\frac{1}{2}}, \tag{3.7}$$

which provides the amplitudes of the two sets of cells of given mode n associated with the two real positive solutions $\alpha_0 = \alpha_{01,2}(x)$ (where $\alpha_{01}^4 \leqslant \frac{1}{3}\lambda C_0 \leqslant \alpha_{02}^4$) of the eigenvalue equation

$$\alpha_0^6 - \lambda C_0(x)\, \alpha_0^2 + 4n^2\pi^2 = 0. \tag{3.8}$$

With the corresponding constants $K_n = K_{n1}, K_{n2}$ related by consideration of the flow near the transition line centred on x_n [given by (2.12)] and use of the boundary conditions at the wall ($x = \frac{1}{2}$) and the additional assumptions stated in § 2, the results of I are modified to give $\alpha_1 = D_{\alpha 1} = 0$ with $K_{n1,2} = 0$ if n is an even integer and otherwise

$$\left.\begin{aligned} |K_{n1}| &= \frac{\gamma^2 c}{4\pi n}\left|\frac{1}{a_2}(\alpha_2\cos[b_2+\delta_n]-b_3\sin[b_2+\delta_n])-\frac{1}{a_1}(\alpha_1\sin[b_1+\delta_n]+b_3\cos[b_1+\delta_n])\right|^{-1}, \\ K_{n2} &= K_{n1}\exp\{i(b_2-b_1-\tfrac{1}{2}\pi)\}, \end{aligned}\right\} \tag{3.9}$$

where

$$\left.\begin{aligned} &\alpha_{1,2} = \alpha_{01,2}(\tfrac{1}{2}), \quad a_{1,2} = |\alpha_{1,2}^4 - \tfrac{1}{3}\lambda C|^{\frac{1}{2}}\alpha_{1,2}^{\beta_1/2\lambda} \\ &b_{1,2} = \gamma^{-\frac{1}{3}}E^{-\frac{1}{3}}\int_{x_n}^{\frac{1}{2}}\alpha_{01,2}\,dx, \quad b_3 = \gamma^{-\frac{1}{3}}2^{\frac{1}{2}}\left(\frac{\lambda C}{3}\right)^{\frac{1}{4}}\cos\tfrac{1}{2}\left\{\tfrac{1}{3}\cos^{-1}\left[\frac{C_0(x_n)}{C}\right]^{\frac{3}{2}}\right\} \end{aligned}\right\} \tag{3.10}$$

and

$$\delta_n = \tan^{-1}\left[\frac{\cos b_1+\{a_1(\alpha_2^4-b_3^4)/a_2(\alpha_1^4-b_3^4)\}\sin b_2}{\sin b_1-\{a_1(\alpha_2^4-b_3^4)/a_2(\alpha_1^4-b_3^4)\}\cos b_2}\right]. \tag{3.11}$$

The physical significance of the regime (3.5) is evident when we note that $\lambda \sim 1, \beta_1 \sim 1$ corresponds to a fluid with Prandtl number $\sigma \sim 1$.

4. The regime $\beta \gg E^{\frac{2}{3}}$ for $\lambda > \lambda_c^{(n)}$

For large values of β_1 we have $a_2 \gg a_1$, so that $K_{n1,2} \propto \alpha_1^{\beta_1/2\lambda}$ and from (3.7)

$$A_{\alpha 1} \propto (\alpha_1/\alpha_{01}(x))^{\beta_1/2\lambda}, \quad A_{\alpha 2} \propto (\alpha_1/\alpha_{02}(x))^{\beta_1/2\lambda} \quad (\beta_1/\lambda \gg 1). \tag{4.1}$$

Since C_0 is monotonic it follows from (3.8) that $\alpha_1 = \alpha_{01}(\frac{1}{2}) \leqslant \alpha_{01}(x) \leqslant \alpha_{02}(x)$ for all $x_n < x \leqslant \frac{1}{2}$, so that the amplitudes of both sets of cells are diminished from their corresponding values when $\beta_1 = 0$. If we fix λ and let $\beta_1/\lambda \to \infty$ (equivalent to $\sigma \to 0$), $A_{\alpha 2}$ vanishes everywhere and $A_{\alpha 1}$ vanishes except near $x = \frac{1}{2}-$, where we have

$$A_{\alpha 1} \propto \exp\left\{\frac{\beta_1}{2\lambda}(\log\alpha_1 - \log\alpha_{01}(x))\right\} \sim \exp\left\{\frac{-(x-\frac{1}{2})^2 c\beta_1}{8(\lambda C - 3\alpha_1^4)}\right\}. \tag{4.2}$$

Here the second exponent follows from the form of the Taylor expansion for $\alpha_{01}(x)$ at $x = \frac{1}{2}$, which is obtained from (3.8). It is interesting to note that this limiting form may be confirmed by a boundary-layer analysis at the wall, demonstrating the shift in emphasis from the transition lines when $\beta \gg E^{\frac{2}{3}}$. We assume that $\beta_1 \gg 1$ and define a scaled horizontal co-ordinate ξ by the relation

$$\beta_1^r \xi = x - \tfrac{1}{2}, \tag{4.3}$$

where the constant r remains unspecified at present. The solution for ψ in the neighbourhood of the wall (but outside the $E^{\frac{1}{3}}$-layer) is written as

$$\psi = E^{\frac{2}{3}}\left\{\frac{c\gamma^2(\frac{1}{2}-x)}{16E^{\frac{1}{3}}} + \gamma^{\frac{1}{3}}\left[\sum_{\alpha_0>0} D_\alpha(\xi, Z, E, \beta_1)\exp\{i\gamma^{-\frac{1}{3}}E^{-\frac{1}{3}}(x-\tfrac{1}{2})\alpha_0\} + \text{c.c.}\right]\right\} + \ldots \quad (E \to 0), \tag{4.4}$$

where the oscillations on the $E^{\frac{1}{3}}$ scale are represented by the parameter α_0, now independent of x. The function D_α remains to be determined and is expanded as $E \to 0$ in the form

$$D_\alpha = D_{\alpha 0}(\xi, Z) + \beta_1^s E^t D_{\alpha 1}(\xi, Z) + \ldots, \tag{4.5}$$

where s and t are unspecified constants.

Substitution of (4.4) into the equations of motion and equating terms of order unity now gives

$$D_{\alpha 0} = A_\alpha(\xi)\sin n\pi Z, \tag{4.6}$$

and the possible values of α_0 are the solutions of (3.8) with $C_0(x)$ replaced by C, for which the series of critical values $\lambda_c^{(n)}$ of λ at which the nth set of roots first reaches the real axis is given by (2.12). The amplitude functions $A_\alpha(\xi)$ are determined by consideration of the equation for $D_{\alpha 1}$. With $\lambda \not\simeq \lambda_c^{(n)}$ for the given value of n the appropriate balance with the first-order derivatives arising from $D_{\alpha 0}$ is obtained by taking

$$r = -\tfrac{1}{2}, \quad s = \tfrac{1}{2}, \quad t = \tfrac{1}{3}, \tag{4.7}$$

and a consistent solution for $D_{\alpha 1}$ satisfying the null boundary conditions at $Z = 0, 1$ can be found only if

$$A_\alpha(\xi) = K_\alpha \exp\{-c\xi^2/8(\lambda C - 3\alpha_0^4)\}, \tag{4.8}$$

where K_α is an arbitrary complex constant to be determined (in (4.9) below) from the boundary conditions at the wall $\xi = 0$.

This result completes the description of the side-wall shear layer when $\beta_1 \gg 1$ for all values of $\lambda \not\simeq \lambda_c^{(n)}$. When $\lambda < \lambda_c^{(1)}$ there are three roots for α_0 with $\operatorname{Im}\alpha_0 < 0$ for all n ($= 1, 2, \ldots$), the solution satisfying the three boundary conditions at the wall decays within a distance $\sim E^{\frac{1}{3}}$ and the value of β is not large enough to influence the flow. If $\lambda_c^{(N+1)} > \lambda > \lambda_c^{(N)}$ for some integer N then for $n \geqslant N+1$ the same argument holds, while for $n = 1, \ldots, N$ we have solutions of the form (4.8) and of the two available real positive values of α_0 we must choose the one which has $\operatorname{Re}(\lambda C - 3\alpha_0^4) > 0$ in order to obtain a solution which decays within a distance $O(\lambda^{\frac{1}{2}}\beta_1^{-\frac{1}{2}})$ of the wall. To leading order the three boundary conditions at the wall are satisfied if the nth mode $\psi^{(n)}$ of the stream function ($n \leqslant N$) is given by

$$\psi^{(n)} = \frac{c\gamma^{\frac{7}{3}}E^{\frac{5}{6}}}{4\pi\alpha_0 n}\sin\frac{\alpha_0(x-\frac{1}{2})}{\gamma^{\frac{1}{3}}E^{\frac{1}{3}}}\exp\left\{\frac{-c\xi^2}{8(\lambda C - 3\alpha_0^4)}\right\}\sin n\pi Z \quad (n = 1, 3, \ldots), \tag{4.9}$$

where α_0 is the appropriate positive root of (3.8) with $\alpha_0^4 < \frac{1}{3}\lambda C$ (i.e. $\alpha_0 = \alpha_1$). This result confirms (4.2) but is independent of the solution in the neighbourhood of the transition lines, the boundary-layer method essentially making the simplifying assumption that zero is a consistent solution in those regions. We also note that the solution which decays within a distance $\sim E^{\frac{1}{3}}$ from the side wall makes no contribution to (4.9) to leading order. For values of λ such that $\lambda \simeq \lambda_c^{(n)}$ the above solutions (4.9) and (3.7) are clearly invalid and a new expansion procedure must be adopted: this is the subject of the next section.

5. The critical regime $\beta \sim E^{\frac{1}{3}}$ for $\lambda \simeq \lambda_c^{(n)}$

Since $C_0'(\frac{1}{2}) = 0$ it follows from (2.12) that when

$$\lambda = \lambda_c^{(n)} + \delta \quad (\delta \ll 1) \tag{5.1}$$

we have
$$\tfrac{1}{2} - x_n \simeq (2C/\lambda c)^{\frac{1}{2}}\delta^{\frac{1}{2}}, \quad C_0'(x_n) \simeq (2Cc/\lambda)^{\frac{1}{2}}\delta^{\frac{1}{2}}, \tag{5.2}$$

and thus the transitional region of § 3, centred on x_n and of horizontal extent

$$\sim E^{\frac{2}{9}}C^{\frac{1}{9}}C_0'(x_n)^{-\frac{1}{3}} \sim E^{\frac{2}{9}}\delta^{-\frac{1}{6}},$$

can no longer be treated independently of the side wall when $\delta \sim E^{\frac{1}{3}}$. For values of λ so close to $\lambda_c^{(n)}$ the cell of mode n is only just generated at the side wall and the solution of § 3 must be reformulated. We therefore define order-one parameters $\tilde{\beta}$ and $\tilde{\delta}$ and an order-one horizontal length scale $\tilde{\theta}$ by

$$\beta = E^{\frac{1}{2}}\gamma^{\frac{7}{6}}\tilde{\beta}, \quad \lambda = \lambda_c^{(n)} + \gamma^{\frac{1}{3}}E^{\frac{1}{3}}\tilde{\delta}, \quad E^{\frac{1}{6}}\gamma^{\frac{1}{6}}\tilde{\theta} = x_n - x, \tag{5.3}$$

where it is anticipated that inertial effects due to the thermal wind will modify the leading-order solution with $\tilde{\delta} \sim 1$ when $\tilde{\beta} \sim 1$.

The transitional solution for the mode n is written in the form

$$\psi^{(n)} = \{\gamma^{\frac{1}{3}}E^{\frac{5}{6}} \sin(n\pi Z)\, A_\alpha(\tilde{\theta}) \exp[i\tilde{\alpha}_0(x-x_n)/\gamma^{\frac{1}{3}}E^{\frac{1}{3}}] + \text{c.c.}\} + \ldots \quad (E \to 0), \tag{5.4}$$

where $\tilde{\alpha}_0 = \alpha_0(x_n) = (2n^2\pi^2)^{\frac{1}{6}}$. Substitution into the full equations of motion shows that the complex function A_α is given by $A_\alpha(\tilde{\theta}) = A(\tilde{\theta})$, where

$$A'' - (\tfrac{1}{4}\theta^2 + a)\, A = 0 \tag{5.5}$$

and

$$\left.\begin{aligned} \theta &= k_1\tilde{\theta} + k_1 k_2 \tilde{\delta}^{\frac{1}{2}} - ik_3\tilde{\beta} \quad (-\tilde{\delta}^{\frac{1}{2}}k_2 < \tilde{\theta} < \infty), \\ a &= -\tfrac{1}{4}k_1^2 k_2^2 \tilde{\delta} + \tfrac{1}{4}k_3^2\tilde{\beta}^2, \end{aligned}\right\} \tag{5.6}$$

with

$$k_1 = (\tilde{\alpha}_0^2 c/2C)^{\frac{1}{4}}, \quad k_2 = 2^{\frac{1}{2}}C/3^{\frac{1}{2}}c^{\frac{1}{2}}\tilde{\alpha}_0^2, \quad k_3 = C^{\frac{3}{4}}c^{\frac{1}{4}}/2^{\frac{1}{4}}6\tilde{\alpha}_0^{\frac{9}{2}}. \tag{5.7}$$

The required solution of (5.5) which decays as $\operatorname{Re}\theta \to \infty$ is

$$A = KU(a,\theta), \tag{5.8}$$

where $U(a,\theta)$ is the parabolic cylinder function (defined, for example, by Abramowitz & Stegun 1964, p. 685) and K is an arbitrary complex constant. In general then, the full solution for the nth mode of the stream function near the outer side wall when $\tilde{\beta}, \tilde{\delta} \sim 1$ which satisfies the no-slip and insulating conditions at the wall may be written in the form

$$\begin{aligned} \psi^{(n)} = \frac{E^{\frac{5}{6}}\gamma^{\frac{7}{3}} c \sin n\pi Z}{4\pi n\tilde{\alpha}_0(R^2+I^2)} \Bigg\{ &\cos\frac{\tilde{\alpha}_0(x-\frac{1}{2})}{\gamma^{\frac{1}{3}}E^{\frac{1}{3}}}[R \operatorname{Im} U(a,\theta) - I \operatorname{Re} U(a,\theta)] \\ &+ \sin\frac{\tilde{\alpha}_0(x-\frac{1}{2})}{\gamma^{\frac{1}{3}}E^{\frac{1}{3}}}[R \operatorname{Re} U(a,\theta) + I \operatorname{Im} U(a,\theta)]\Bigg\} + O(E) \quad (E \to 0), \end{aligned} \tag{5.9}$$

where $R, I = \operatorname{Re}, \operatorname{Im}\{U(a, -ik_3\tilde{\beta})\}$, although this solution may become invalid at certain discrete values of λ (see § 6 below). The solution which decays exponentially within a distance $\sim E^{\frac{1}{3}}$ from the side wall makes no contribution to leading order as $E \to 0$ and thus the behaviour of the cells near $\lambda = \lambda_{(c)}^n$ is solely dependent upon the properties of the function $U(a,\theta)$.

First, when $\tilde{\beta} = 0$ and $\tilde{\delta} \sim 1$ we have $a = -\frac{1}{4}k_1^2k_2^2\tilde{\delta}$, $I = 0$, $R = U(a,0)$ and θ is purely real. As $\tilde{\delta} \to \infty$,

$$U(a,\theta) \sim 2^{-\frac{1}{4}-\frac{1}{2}a}\,\Gamma(\tfrac{1}{4}-\tfrac{1}{2}a)\left(\frac{t}{\zeta^2-1}\right)^{\frac{1}{4}} \operatorname{Ai}(t), \tag{5.10}$$

where

$$t = (4|a|)^{\frac{2}{3}}\tau, \quad \theta = 2|a|^{\frac{1}{2}}\zeta, \quad \tau = \mp(\tfrac{3}{2}\theta_1)^{\frac{2}{3}} \quad (\zeta \gtrless 1), \tag{5.11}$$

and

$$\theta_1 = \left\{\begin{aligned} &\tfrac{1}{4}\cos^{-1}\zeta - \tfrac{1}{4}\zeta(1-\zeta^2)^{\frac{1}{2}} \quad (\zeta \leqslant 1), \\ &\tfrac{1}{4}\zeta(\zeta^2-1)^{\frac{1}{2}} - \tfrac{1}{4}\cosh^{-1}\zeta \quad (\zeta \geqslant 1). \end{aligned}\right\} \tag{5.12}$$

This represents an Airy function Ai centred on $\zeta = 1$ (i.e. $\theta = k_1 k_2 \tilde{\delta}^{\frac{1}{2}}$, i.e. $x = x_n$) and decays within a distance $\sim |a|^{-\frac{1}{3}}$ (i.e. $\theta \sim \tilde{\delta}^{-\frac{1}{6}}$, i.e. $x - x_n \sim E^{\frac{1}{3}}\delta^{-\frac{1}{6}}$) on either side of this line. Thus as $\tilde{\delta} \to \infty$ the transitional region spreads from the wall, developing a region of dominant amplitude near x_n with oscillatory behaviour in the region $x > x_n$, the solutions in these regions matching precisely with those of § 3 as $\delta \to 0$.

Inertial effects upon the solution may be judged from an asymptotic expansion of U near the wall as $\tilde{\delta} \to \infty$ for general values of $\tilde{\beta} \sim 1$. In this case, if we write

$$\theta = -i\tilde{\beta}k_3 + \bar{\theta}, \tag{5.13}$$

so that $\bar{\theta}$ is real and varies from zero at the wall to infinity as $\operatorname{Re}\theta \to \infty$, then

$$U(a,\theta) \propto \exp\{\tfrac{1}{2}ik_1k_2\tilde{\delta}^{\frac{1}{2}}\bar{\theta} - \tilde{\beta}k_3\bar{\theta}^2/4k_1k_2\tilde{\delta}^{\frac{1}{2}}\} \quad (\tilde{\delta} \to \infty), \tag{5.14}$$

where terms of exponentially small order are neglected and in addition to $\tilde{\delta} \gg 1$, it is assumed that $\bar{\theta}^2, \tilde{\beta}\bar{\theta} \ll \tilde{\delta}$. The solution therefore represents a boundary layer near the wall where $\bar{\theta} \sim \tilde{\beta}^{-\frac{1}{2}}\tilde{\delta}^{\frac{1}{4}}$ in which the solution oscillates rapidly with wavelength $\sim \tilde{\delta}^{-\frac{1}{2}}$, provided that

$$\tilde{\delta}^{-\frac{1}{2}} \ll \tilde{\beta} \ll \tilde{\delta}^{\frac{3}{2}}. \tag{5.15}$$

The upper limit expresses the fact that the wavelength of the oscillation is small on the boundary-layer length scale whilst the lower limit ensures that the boundary layer does not extend as far as the transition line situated at $\bar{\theta} = k_1k_2\tilde{\delta}^{\frac{1}{2}}$. Here the detailed structure of the flow may be determined but from (5.14) it follows that for $\tilde{\delta} \gg 1$ it is exponentially small and thus insignificant in comparison with that near the wall even for small values of the inertial parameter $\tilde{\beta}$. If $\tilde{\beta} \sim \tilde{\delta}^{-\frac{1}{2}}$ the boundary-layer solution fails although the oscillations of wavelength $\sim \tilde{\delta}^{-\frac{1}{2}}$ remain and the solution is a perturbation of the $\tilde{\beta} = 0$ solution when $\tilde{\beta} \ll \tilde{\delta}^{-\frac{1}{2}}$.

For large values of $\tilde{\beta}$ and general values of $\tilde{\delta} \sim 1$ the solution may be expressed in terms of the Hankel function $H^{(1)}_{\frac{1}{3}}$, and provided that $\tilde{\beta} \gg \tilde{\delta}^{\frac{1}{2}}, 1$

$$U(a,\theta) \propto \left(\bar{\theta} - \frac{ik_1^2k_2^2\tilde{\delta}}{2k_3\tilde{\beta}}\right)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}\left(\left[\frac{1+i}{3}\right]k_3^{\frac{1}{2}}\tilde{\beta}^{\frac{1}{2}}\left\{\bar{\theta} - \frac{ik_1^2k_2^2\tilde{\delta}}{2k_3\tilde{\beta}}\right\}^{\frac{3}{2}}\right), \tag{5.16}$$

where $\bar{\theta}$ is defined by (5.13). This represents a solution which decays within a boundary layer of thickness $\bar{\theta} \sim \tilde{\beta}^{-\frac{1}{3}} (\ll 1)$ when $\tilde{\delta} \sim \tilde{\beta}^{\frac{2}{3}}$ (the effect of $\tilde{\delta}$ being negligible to leading order if $\tilde{\delta} \ll \tilde{\beta}^{\frac{2}{3}}$) whilst if $\tilde{\delta} \gg \tilde{\beta}^{\frac{2}{3}}$ it is easily verified that we recover the boundary-layer behaviour (5.14) with the region of decay extending from $\sim \tilde{\beta}^{-\frac{1}{3}}$ to a distance $\sim \tilde{\beta}^{-\frac{1}{2}}\tilde{\delta}^{\frac{1}{4}}$ from the wall.

From the above analysis we see that if $\tilde{\beta} \gg \tilde{\delta}^{-\frac{1}{2}}$ the solution develops a boundary-layer structure [of the form (5.14)] near the wall, as $\tilde{\delta} \to \infty$, in line with the limiting behaviour predicted for $a_2 \gg a_1$ (which is equivalent to $\beta_1 \gg (\lambda - \lambda_c^{(n)})^{-\frac{1}{2}} \sim \delta^{-\frac{1}{2}}$, i.e. $E^{\frac{1}{3}}\tilde{\beta} \gg E^{\frac{1}{3}}\tilde{\delta}^{-\frac{1}{2}}$) at the beginning of § 4. Of course the boundary layer does not always develop because we cannot take $\tilde{\delta}$ arbitrarily large; the solution must become invalid before $\tilde{\delta} \sim E^{-\frac{1}{3}}$ (i.e. $\delta \sim 1$). Essentially the formation of the boundary layer is a function of both β and $\lambda - \lambda_c^{(n)}$, expressible as a single parameter $\tilde{\beta}\tilde{\delta}^{\frac{1}{2}}$ when $\lambda - \lambda_c^{(n)} \ll 1$. In general the limit $\tilde{\delta} \to \infty$ with $\tilde{\beta}\tilde{\delta}^{\frac{1}{2}} \sim 1$ produces the full cellular structure of § 3, valid for all $\beta \sim E^{\frac{1}{2}}$ when $\lambda - \lambda_c^{(n)} \sim 1$, but if $\tilde{\delta} \to \infty$ with $\tilde{\beta}\tilde{\delta}^{\frac{1}{2}} \to \infty$ this becomes the boundary-layer solution of § 4, which is then valid for $\beta \gg E^{\frac{1}{2}}$ when $\lambda - \lambda_c^{(n)} \sim 1$, while if $\tilde{\delta} \to \infty$ with $\tilde{\beta}\tilde{\delta}^{\frac{1}{2}} \to 0$ it is simply a perturbation of the limiting solution when $\tilde{\beta} = 0$.

6. Resonance

In Chandrasekhar's (1961, chap. 2) study of the stability of an infinite rapidly rotating thermal layer heated from below, bifurcations of the basic state always appear in the form of steady convection cells as the Rayleigh number increases provided that the Prandtl number exceeds the critical value of 0·676.... In a finite geometry we must also expect bifurcations to occur although they may be inhibited by the container walls, leading to an increased critical Rayleigh number, and also occur in a linear form only at discrete intervals: essentially when the cells are of just the right dimensions to fit into the container and satisfy the null conditions at the walls. These aspects have been considered numerically in the rotating case by Homsy & Hudson (1971) and analytically in the non-rotating case by Drazin (1975) and Segel (1969). If the cells are actually forced by non-zero boundary conditions at the walls, as in the problem considered here, then resonance is liable to occur when their frequency coincides with one of the natural spatial frequencies of the container. In this section we investigate the effect of the thermal wind on the occurrence of resonance by determining the location of the resonant states for values of $\tilde{\beta}$ in the range zero to infinity under the approximate formulation of §§2–5.

First, if $\lambda - \lambda_c^{(n)} = \delta$ with $\delta \sim 1$ and $\beta_1 \sim 1$ and n is an odd integer, the resonant states are determined by the solutions of the equation

$$-\frac{\alpha_1}{a_1}\sin(b_1+\delta_n)+\frac{\alpha_2}{a_2}\cos(b_2+\delta_n)-\frac{b_3}{a_1}\cos(b_1+\delta_n)-\frac{b_3}{a_2}\sin(b_2+\delta_n) = 0, \tag{6.1}$$

since then K_{n1}, as given by (3.9), becomes infinite. Physically, this means that the vertical component of velocity in the convection cells increases from the value of order $E^{\frac{1}{2}}$ imposed by the side-wall boundary condition to a significantly larger value measured by an appropriate power of E in the neighbourhood of any value of λ at which (6.1) is satisfied. The precise amount by which the amplitude rises will ultimately depend either upon nonlinear effects, or upon the higher-order terms in the expansion (2.8) in powers of E ($D_{\alpha 0}$ being the leading term). The latter effect is possible because once the leading-order term is sufficiently large, the amplitude of the second-order term will become comparable with the magnitude of the side-wall boundary condition ($\sim E^{\frac{1}{2}}$) and thus will play a significant role. If n is an even integer the condition (6.1) corresponds to the existence of an eigensolution in which the value of K_{n1} is arbitrary.

Analytic solutions of (6.1) are possible in several limiting cases. If $\lambda \gg n^{\frac{2}{3}}$, so that

$$\left.\begin{aligned}&\alpha_1 \simeq 2n\pi/\lambda^{\frac{1}{2}}C^{\frac{1}{2}}, \quad \alpha_2 \simeq b_3 \simeq \lambda^{\frac{1}{2}}C^{\frac{1}{2}},\\ &a_1 \simeq (\tfrac{1}{3}\lambda C)^{\frac{1}{2}}\alpha_1^{1/2\sigma}, \quad a_2 \simeq (\tfrac{2}{3}\lambda C)^{\frac{1}{2}}\alpha_2^{1/2\sigma}, \quad x_n = O(n^{\frac{2}{3}}/\lambda)\end{aligned}\right\} \tag{6.2}$$

and, from (3.11),

$$\cos(b_1+\delta_n) = O(n^2/\lambda^{\frac{3}{2}}), \tag{6.3}$$

condition (6.1) reduces to

$$\frac{\alpha_2}{a_2}\cos(b_2+\delta_n)-\frac{b_3}{a_2}\sin(b_2+\delta_n)-\frac{\alpha_1}{a_1}\sin(b_1+\delta_n) = O\left(\frac{b_3 n^2}{a_1\lambda^{\frac{3}{2}}}\right), \tag{6.4}$$

and resonance depends upon the relative magnitudes of the terms α_2/a_2, b_3/a_2, α_1/a_1 and b_3/a_1. If $\sigma > \frac{1}{2}$ the significant terms are the first two and the criterion for resonance reduces to

$$b_2 - b_1 = m\pi - \tfrac{1}{4}\pi \tag{6.5}$$

for any integer m. If $\sigma = \frac{1}{2}$ the third term is also significant, leading to the slightly modified condition

$$b_2 - b_1 = m\pi - \tfrac{7}{12}\pi, \tag{6.6}$$

while if $\sigma < \frac{1}{2}$ the third term is dominant and thus resonance is impossible. Since $b_2 \gg b_1$ and $x_n \simeq 0$ the above conditions become

$$\lambda^{\frac{1}{4}}\gamma^{-\frac{1}{3}}E^{-\frac{1}{3}}\int_0^{\frac{1}{2}} C_0^{\frac{1}{4}}\,dx = \begin{Bmatrix}(m-\frac{1}{4})\pi & (\sigma > \frac{1}{2}) \\ (m-\frac{7}{12})\pi & (\sigma = \frac{1}{2})\end{Bmatrix} \tag{6.7}$$

to leading order and for given γ, C_0 and E these are satisfied at an infinite number of values of λ with $m \sim E^{-\frac{1}{3}}$. In both cases successive states for a given n are separated by a difference

$$4\pi\lambda^{\frac{3}{4}}\gamma^{\frac{1}{3}}E^{\frac{1}{3}}\bigg/\int_0^{\frac{1}{2}} C_0^{\frac{1}{4}}\,dx \tag{6.8}$$

in the value of λ, only slight changes being necessary to 'fit' additional cells into the annulus. Of course the corresponding change in the conventional Rayleigh number R between successive resonance states is large, being of order E^{-1} as $E \to 0$.

Solutions of (6.1) may be found for general values of λ in the limit as $\sigma \to 0$ ($\beta_1 \to \infty$). Suppose that $\alpha_2 - \alpha_1 \sim \Delta$, so that $(\alpha_2/\alpha_1)^{1/2\sigma} \gg 1$ if $\sigma \ll \Delta$. Therefore $a_2 \gg a_1$ if $\sigma \ll \Delta$ and then (3.11) reduces to $\cos(b_1 + \delta_n) = 0$ and the criterion for resonance to $\sin(b_1 + \delta_n) = 0$. Thus resonance is impossible if $\sigma \ll \alpha_2 - \alpha_1$. For values of $\delta \sim 1$ we have $\alpha_2 - \alpha_1 \sim 1$ and so resonance is impossible if $\sigma \ll 1$. This is consistent with the lower limit of $\sigma = \frac{1}{2}$ when $\lambda \gg n^{\frac{4}{3}}$ and is confirmed explicitly by the boundary-layer solution (4.9), while as $\delta \to 0$ these results hold if $\sigma \ll \delta^{\frac{1}{2}}$.

The occurrence of resonance at very small values of δ (including the first resonance as λ increases beyond $\lambda_c^{(n)}$) depends upon the properties of the parabolic cylinder function $U(a,\theta)$, this solution replacing the solution (2.8), which gives rise to the condition (6.1) when $\tilde{\delta} = E^{-\frac{1}{3}}\gamma^{-\frac{1}{3}}\delta \sim 1$. In general we see from (5.9) that resonance occurs when $U(a, -ik_3\tilde{\beta}) = 0$, so that if $\tilde{\beta} = 0$ it takes place when $U(a,0) = 0$; i.e. when

$$\tilde{\delta} = 6(4m+3)\,c^{\frac{1}{2}}\tilde{\alpha}_0^3/2^{\frac{1}{2}}C^{\frac{3}{2}} \quad (m = 0, 1, 2, \ldots), \tag{6.9}$$

the first resonance occurring at a value of λ which is $18\gamma^{\frac{1}{3}}c^{\frac{1}{2}}\tilde{\alpha}_0^3E^{\frac{1}{3}}/2^{\frac{1}{2}}C^{\frac{3}{2}}$ in excess of $\lambda_c^{(n)}$. On the other hand, if $\tilde{\beta}$ is small (but non-zero) resonance will occur only if

$$U(a,0) - ik_3\tilde{\beta}U'(a,0) = 0 \tag{6.10}$$

when $\tilde{\beta} \gg E^{\frac{1}{6}}$ and since $a = -\frac{1}{4}k_1^2k_2^2\tilde{\delta} + O(\tilde{\beta}^2)$ this can never be achieved, although clearly the amplitude will rise to a value $\sim \tilde{\beta}^{-1}$ ($\gg 1$) when $\tilde{\delta}$ takes any one of the values given by (6.9).

If $\tilde{\beta} \sim E^{\frac{1}{6}}$ the second term in the expansion (6.10) is of the same order as the second-order terms in the expansion of the solution (5.9) in powers of E, which must therefore be taken into account. The above argument shows that in general we may expect the amplitude of the vertical velocity in the cells to increase from order $E^{\frac{1}{2}}$ to order $E^{\frac{1}{3}}$ near the resonance positions (6.9), although as mentioned above, a full investigation of the second-order terms is required to provide a detailed description of the behaviour of the solution near these positions. It is hoped to report on this aspect of the problem in a future paper. We note, however, that resonance does not occur as $\tilde{\delta} \to \infty$ even for small values of $\tilde{\beta}$ if $\tilde{\beta}\tilde{\delta}^{\frac{1}{2}} \gg 1$, the expansion (6.10) being invalid owing to the rapid oscillation of the boundary-layer solution (5.14).

Finally we consider the limit $\tilde{\beta} \to \infty$. From (5.16) we see that resonance then depends upon the zeros of the Hankel function $H^{(1)}_{\frac{1}{3}}(e^{-\frac{1}{2}i\pi}\tilde{\Delta})$, where $\tilde{\Delta} = \tilde{\delta}^{\frac{3}{2}} k_1^3 k_2^3 / 3k_3 \tilde{\beta} 2^{\frac{3}{2}}$. However

$$H^{(1)}_{-\frac{1}{3}}(e^{-\frac{1}{2}i\pi}\tilde{\Delta}) \propto K_{\frac{1}{3}}(\tilde{\Delta}) + i\{3^{\frac{1}{2}} K_{\frac{1}{3}}(\tilde{\Delta}) + 2\pi I_{\frac{1}{3}}(\tilde{\Delta})\}, \tag{6.11}$$

where $K_{\frac{1}{3}}$ and $I_{\frac{1}{3}}$ are the modified Bessel functions, and since $K_{\frac{1}{3}}(\tilde{\Delta})$ has no zeros for $|\arg\tilde{\Delta}| \leqslant \frac{1}{2}\pi$ there is no resonance for any $\tilde{\Delta} \geqslant 0$. Thus resonance does not occur for any $\tilde{\delta}$ such that $0 < \tilde{\delta} < \infty$, while for larger values of λ the boundary-layer solution (4.9) confirms that resonance is completely eliminated for all values of λ when $\tilde{\beta} \gg 1$.

7. Discussion

We have studied the modifications to patterns of steady cellular convection which occur as the effect of an interior thermal wind becomes significant. First, if $\lambda < \lambda_c^{(1)}$ there are no cells and the flow is modified by nonlinear inertial effects if a thermal Rossby number $\beta \sim 1$. However if $\lambda > \lambda_c^{(1)}$ the structure of the cells that are present is changed if

$$\beta_1/\lambda = \beta\gamma^{\frac{4}{3}}\lambda^{-1}E^{-\frac{2}{3}} = \sigma^{-1} \sim 1. \tag{7.1}$$

The change takes the form of a damping of the amplitudes of the cells, which still retain their original transition lines throughout the outer half of the annulus. In the limit $\beta_1/\lambda \to \infty$ (i.e. $\sigma \to 0$) the cells are damped out almost completely and only one of each set corresponding to a given vertical mode n remains of any significance and this only in the immediate neighbourhood of the outer wall, where the maximum negative temperature gradient occurs. The transition lines become insignificant in comparison with the flow near the wall, the boundary-layer analysis of § 4 showing that when $\beta_1/\lambda \gg 1$ the five arbitrary constants which appear (for given n) in the solution when $\beta_1 \sim 1$ (two of which are determined from consideration of the flow near transition) are reduced in number to three and are completely determined from the conditions at the wall. However the damping of the cells is effected by a smooth transition in which, provided that $\lambda/\beta_1 = \sigma \ll 1$, their extent is limited to a maximum distance of order

$$(\lambda/\beta)^{\frac{1}{2}} = \sigma^{\frac{1}{2}} \tag{7.2}$$

from the outer side wall. These steady cells are therefore likely to be of least significance in fluids with low Prandtl numbers.

In an infinite layer convection cells which oscillate in time can occur at values of λ less than $\lambda_c^{(1)}$ if the Prandtl number is less than 0·676... (Chandrasekhar 1961, chap. 2), overstability being preferred to the exchange of stabilities in this situation. In the present paper we have discussed only steady linear solutions of the system and an investigation of the influence of the side wall and the horizontal temperature gradient on the stability characteristics of the flow is required to determine whether, for sufficiently small Prandtl numbers, either nonlinear or time-dependent oscillatory motions are as important as the steady component of the flow described here. One possibility is that oscillatory disturbances will be convected away from the side wall with the characteristic group velocity of the system, and subsequently damped out in the stably stratified interior region.

As the value of λ passes through $\lambda_c^{(n)}$ and a cell of mode n is generated, the horizontal variation of its amplitude is expressed as a parabolic cylinder function, inertial effects being significant if $\beta \gtrsim E^{\frac{1}{2}}$ and the solution eventually developing into either the cellular regime of § 3 if $\beta \sim E^{\frac{2}{3}}$ or the boundary-layer regime of § 4 if $\beta \gg E^{\frac{2}{3}}$, the latter being formed when $(\lambda - \lambda_c^{(n)}) E^{-\frac{1}{3}} \gg 1$ if $\beta E^{-\frac{1}{2}} \gg [(\lambda - \lambda_c^{(n)}) E^{-\frac{1}{3}}]^{-\frac{1}{2}}$. From (5.15) the boundary-layer solution remains consistent when $\lambda - \lambda_c^{(n)} \sim 1$ only if $E^{\frac{1}{6}} \ll \beta E^{-\frac{1}{2}} \ll E^{-\frac{1}{2}}$, the lower limit ensuring that the layer is thin and the upper limit,

$$\beta \ll 1, \tag{7.3}$$

that the horizontal scale of decay is larger than the oscillatory wavelength. This upper limit is confirmed by the expansion (4.5) for D_α [with the values of s and t given by (4.7)], which breaks down when $\beta_1 \sim E^{-\frac{2}{3}}$. The horizontal scale of decay (7.2) is then of the same order of magnitude as the $E^{\frac{1}{3}}$ side-wall layer, and the thermal Rossby number $\beta \sim 1$. In this situation inertial modifications to the entire conductive solution must be taken into account and the problem in the interior is nonlinear.

Taking the limit of the present solution as $\beta_1 \to \infty$ indicates that if β is small (but order one) a consistent solution will exist in boundary layers along the inner and outer walls of the annulus which will provide the necessary local adjustment of the flow to the conditions at the wall and also decay into the geostrophic interior within a distance $O(E^{\frac{1}{3}})$. On the other hand once β is sufficiently large the shearing in the core can lead to instability even in regions of positive temperature gradient, depending upon the attainment of a critical Richardson-like number in the fluid and the difference of the Prandtl number from unity. The critical wavelength at instability was determined by Walton (1975) in the case of a vertically bounded system of small aspect ratio (equivalent to the assumption that $\beta \gg 1$ in the present study), thus extending the original inviscid result of Stone (1966). McIntyre (1970) has also considered the problem in an unbounded viscous fluid, equivalent to finite values of β and E in the present study. His analysis predicts an infinite critical wavelength on the viscous length scale $\sim E^{\frac{1}{2}}$ (showing that for $E \sim 1$ the marginal instability modes in any container will depend on the container geometry in an essential way), and although his general results are confirmed by the experiments of Baker (1971) and by Walton's analysis in the limit $\beta \to \infty$, $E \to 0$, the effect that is of interest in the present context, namely that of lateral bounding walls, remains to be determined.

Although the present study has relied heavily on a simplification of the interior temperature field, the results may be extended to general stratification fields [e.g. (2.7)] in a straightforward manner. If $T_0(x, z) = -\gamma P(x, Z)$, $\beta_1 \sim 1$ and $\lambda \gg n^{\frac{4}{3}} \gg 1$ an analytic solution may be found using the WKB method (see I) in the form

$$D_{\alpha 0} = \frac{A_\Omega(x)}{(P_Z - \Omega_0^4)^{\frac{1}{4}}} \sin\left\{\tfrac{1}{2}\lambda^{\frac{3}{4}}|\Omega_0| \int_Z^{Z_0} (P_Z - \Omega_0^4)^{\frac{1}{2}}\, dZ + \tfrac{1}{4}\pi\right\} \quad (Z < Z_0), \tag{7.4}$$

where the two sets of vertical transition points $Z_0 = Z_{01}, Z_{02}$ (above which the solution is exponentially small), eigenvalues $\Omega_0 \equiv \lambda^{-\frac{1}{4}}\alpha_0 = \Omega_1, \Omega_2$ and amplitude functions $A_\Omega = A_{\Omega 1}, A_{\Omega 2}$ are now given by

$$Z_{01} \simeq 1, \quad \Omega_1 \simeq \frac{2\pi(n - \frac{1}{4})}{\lambda^{\frac{1}{4}} \int_0^1 P_Z^{\frac{1}{2}}\, dZ}, \quad |A_{\Omega 1}| = K_{n1} \exp\left\{-\left(\tfrac{1}{4} - \frac{\beta_1}{4\lambda}\right) \int_0^x \frac{h(x')\, dx'}{g(x')}\right\}, \tag{7.5}$$

$$Z_{02} \simeq \frac{\{3\pi(n - \frac{1}{4})\}^{\frac{2}{3}}}{\lambda^{\frac{1}{2}}\{-P_{ZZ}(x, 0)\}^{\frac{1}{3}}\{P_Z(x, 0)\}^{\frac{1}{6}}}, \quad \Omega_2 \simeq P_Z(x, 0)^{\frac{1}{4}}, \quad |A_{\Omega 2}| = \frac{K_{n2}\{-P_{ZZ}(x, 0)\}^{\frac{1}{3}}}{\{P_Z(x, 0)\}^{\frac{11}{24} + \beta_1/2\lambda}}, \tag{7.6}$$

where
$$g = \int_0^1 P_Z^{\frac{1}{2}}\,dZ, \quad h = \int_0^1 \frac{P_{xZ}\,dZ}{P_Z^{\frac{1}{2}}}. \tag{7.7}$$

As $\beta_1\lambda^{-1} \to \infty$ the amplitude of the second set (7.6) becomes exponentially small, as does that of the first set, except in the immediate neighbourhood of the outer wall. It may be confirmed by a boundary-layer analysis (equivalent to that of § 4) that the cells are confined to a boundary-layer region $-\xi = \lambda^{-\frac{1}{2}}\beta_1^{\frac{1}{2}}(\frac{1}{2}-x) \sim 1$, where the solution corresponding to (7.6) must be discarded (since $P_{ZZZ}(\frac{1}{2}, Z) > 0$) and

$$\psi^{(n)} \simeq E^{\frac{5}{6}}\gamma^{\frac{1}{3}}\,\mathrm{Re}\left\{\frac{A_{\Omega 1}(\xi)}{(P_Z(\frac{1}{2},Z)-\Omega_1^4)^{\frac{1}{4}}}\exp\{i\gamma^{-\frac{1}{3}}\lambda^{\frac{1}{4}}\Omega_1 E^{-\frac{1}{3}}(x-\tfrac{1}{2})\}\right.$$
$$\left.\times\sin\left\{\frac{\lambda^{\frac{3}{4}}}{2}|\Omega_1|\int_Z^1 (P_Z(\tfrac{1}{2},Z)-\Omega_1^4)^{\frac{1}{2}}\,dZ+\tfrac{1}{4}\pi\right\}\right\}, \tag{7.8}$$

where Ω_1 is given by (7.5) evaluated at $x = \frac{1}{2}$,

$$|A_{\Omega 1}| = K_{\Omega 1}\exp\left\{-\frac{\bar{h}\xi^2}{8\bar{g}}\right\}, \quad \bar{h} = \int_0^1 \frac{P_{ZZZ}(\frac{1}{2},Z)\,dZ}{P_Z^{\frac{1}{2}}(\frac{1}{2},Z)}, \quad \bar{g} = \int_0^1 P_Z^{\frac{1}{2}}(\tfrac{1}{2},Z)\,dZ, \tag{7.9}$$

and $K_{\Omega 1}$ is a constant which depends upon the boundary conditions at the wall. The assumption that $\lambda \gg n^{\frac{4}{3}}$ excludes the possibility that a cell of mode n is about to develop at the side wall and so there is no counterpart of the solution (5.8) in this analysis.

In fact the variation with ξ of the solution in the boundary layer is always of the form given by (7.9) for *any* appropriate value of λ ($> \lambda_c^{(n)}$) and any given value of n. However the integrands in the formulae (7.9) for $\bar{h}$ and $\bar{g}$ must be replaced by

$$P_{ZZZ}(\tfrac{1}{2},Z)\,B_\alpha^2, \quad (2P_Z(\tfrac{1}{2},Z)-6\alpha^4\lambda^{-1})\,B_\alpha^2, \tag{7.10}$$

where α and B_α are the real eigenvalues and solutions of the system (2.10), in which we write α for α_0, B_α for $D_{\alpha 0}$ and $-\gamma P(\frac{1}{2}, Z)$ for T_0, respectively. Similar remarks apply to the solution of mode n for values of λ in the immediate neighbourhood of $\lambda_c^{(n)}$, the x variation of the parabolic cylinder function U also being appropriate in the case of a general stratification field.

These extensions of the theory suggest that the phenomenon of resonance described for constant vertical stratification fields in § 6 is a real one. When the effect of the thermal wind is negligible ($\beta \ll E^{\frac{2}{3}}$) the first resonance occurs (as λ increases) at a value of λ of order $E^{\frac{1}{3}}$ in excess of $\lambda_c^{(1)}$, the value at which the first forced convection cells are generated. Although further resonances then follow in rapid succession at regular intervals $O(E^{\frac{1}{3}})$ as λ increases, a nonlinear analysis will be required to determine precisely how the amplitude of the cells evolves in the neighbourhood of the first resonance as the instability takes over. Thermal-wind effects, however, not only damp the cells when they appear but also reduce the possibility of resonance, for it is shown in § 6 that this becomes impossible for large values of λ if $\sigma < \frac{1}{2}$ and if $\lambda-\lambda_c^{(n)} \sim 1$ is impossible (for the given mode n) if $\sigma \ll 1$. The properties of the parabolic cylinder function determine the onset of resonance when $\lambda \simeq \lambda_c^{(n)}$ and confirm that it does not occur for any value of λ when $\beta \gg E^{\frac{1}{2}}$.

For even modes n there is no forcing at the side wall and thus the occurrence of resonance is replaced by the existence of an eigensolution with arbitrary amplitude which satisfies null conditions at the walls. When $\beta \gg E^{\frac{1}{2}}$ no such solutions exist, but this does not indicate that the system is then stable since a full investigation of the stability of the system requires consideration of the possibility that instability may evolve through oscillatory (rather than steady) convection. A related question is the one of nonlinear finite amplitude effects. In the present study we have restricted attention to the behaviour of the steady linear solution forced (through the side-wall boundary condition) by the circulation of the basic flow in the annulus. The behaviour of this solution suggests that, if the Prandtl number is sufficiently high for overstability to be ruled out, a smooth transition may occur to a stable finite amplitude branch in the neighbourhood of the first resonance position, the linear solution becoming unstable beyond this point. Even in cases where the linear solution does not resonate there is the possibility that nonlinear branches may evolve through bifurcations similar to those which occur in the non-rotating Bénard problem discussed by Segel (1969).

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REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions*. Dover.

BAKER, D. J. 1971 *Science* **172**, 1029.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

DANIELS, P. G. 1976 *Geophys. Fluid Dyn.* **7**, 297.

DANIELS, P. G. & STEWARTSON, K. 1977 *Math. Proc. Camb. Phil. Soc.* **81**, 325.

DEFANT, A. 1961 *Physical Oceanography*, vol. I. Pergamon.

DRAZIN, P. G. 1975 *Z. angew. Math. Phys.* **26**, 239.

HIDE, R. & MASON, P. J. 1975 *Adv. in Phys.* **24**, 47.

HOMSY, G. M. & HUDSON, J. L. 1971 *J. Fluid Mech.* **45**, 353.

MCINTYRE, M. E. 1970 *Geophys. Fluid Dyn.* **1**, 19.

RIVAS, E. K. DE 1973 *J. Atmos. Sci.* **30**, 763.

SEGEL, L. A. 1969 *J. Fluid Mech.* **38**, 203.

STONE, P. H. 1966 *J. Atmos. Sci.* **23**, 390.

WALTON, I. C. 1975 *J. Fluid Mech.* **68**, 757.